J. Fluid Mech. (1976), *vol.* 76, *part* 4, *pp.* 819–828
Printed in Great Britain

Double boundary layers in standing interfacial waves

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(Received 17 July 1975 and in revised form 2 July 1976)

The double-boundary-layer theory of Stuart (1963, 1966) and Riley (1965, 1967) is employed to investigate the mass transport velocity due to two-dimensional standing waves in a system comprising two homogeneous fluids of different densities and viscosities. The most important double-boundary-layer structure occurs in the neighbourhood of the oscillating interface, and the possible existence of jet-like motions is envisaged at nodal positions, owing to the nature of the mean flows in the layers. In practice, the magnitude of the mass transport velocity can be a significant fraction of that of the primary, oscillatory velocity.

1. Introduction

Recently, Dore (1970, 1973) has considered the mass transport velocity due to surface and internal interfacial waves in a system comprising two layers of homogeneous fluid of different densities ρ and kinematic viscosities ν. Such velocity is in the Lagrangian sense, and is averaged over a complete period $2\pi/\sigma$ of the motion. The calculations followed the earlier theory for surface waves of Longuet-Higgins (1953) who considered the oscillatory laminar boundary layers, of thickness $O([\nu/\sigma]^{\frac{1}{2}})$, adjacent to the bottom and the oscillating free surface. Whereas the vorticity due to the primary, oscillatory motion is confined to these layers, the extent to which the mean vorticity field penetrates the remainder of the fluid depends, *inter alia*, on the parameters

$$\alpha = ak, \quad \epsilon^{-1} = (\sigma/\nu k^2)^{\frac{1}{2}}, \quad kh.$$

These represent, respectively, the maximum wave slope (assumed $\ll 1$), the square root of a characteristic wave Reynolds number (assumed $\gg 1$) and the ratio of fluid depth h to wavelength $2\pi/k$, for a wave of amplitude a.

The validity of the above-mentioned work requires the waves to be of sufficiently small amplitude. In fact, when $\alpha \gg \epsilon^n$, where $n > 0$ depends on the particular wave under consideration, convection of mean vorticity (neglected in the above calculations) is very important. Indeed, beyond the oscillatory layers, Longuet-Higgins (1953) attempted to calculate the mass transport velocity for these large amplitude surface waves by means of a theory of inviscid rotational flow, in which only the convection term is retained in the equation governing the mean motion. Reasons why the attempt was unsuccessful have been indicated by Stuart (1963, 1966) and Riley (1965, 1967), who studied mean motions in boundary layers adjacent to solid bodies oscillating in unbounded fluid which is otherwise at rest. They demonstrated that, for sufficiently large amplitudes, the mean vorticity is mostly confined within a second (outer) boundary layer which is thicker than the oscillatory layers, and that convection and viscous diffusion of mean vorticity are in balance throughout this outer layer.

The double-boundary-layer theory of Stuart and Riley has been employed by Dore (1976) in a consideration of mean motion due to standing surface waves in homogeneous fluid of finite uniform depth. When $\alpha \gg \epsilon$, the horizontal component of the mass transport velocity, which is $O(\alpha^2)$, decays to zero through an outer bottom boundary layer of thickness $O(\epsilon/\alpha)$. Moreover, near the bottom, upward vertical 'jet-like' motions are predicted beneath *anti-nodal* positions, where the mean flows in the outer boundary layer in two adjacent cells, of width one-quarter of a wavelength, collide. In the absence of any interaction between the induced mean flows near the bottom and free-surface boundary layers, it is shown that no outer boundary layer adjacent to the free surface (assumed clean) is required. The region near the free surface was not considered by Mei, Liu & Carter (1972), who had, however, previously deduced the jet-like structure near the bottom.

In the present work, it is intended to make use of the double-boundary-layer theory of Stuart and Riley in the case of standing internal waves on the interface between two immiscible homogeneous fluids of infinite, or possibly finite, depth. Although it is possible to give a formal development of the required theory in terms of asymptotic expansions and limit processes, as systematically illustrated in the work of Riley (1967), we shall mostly be content to extract and investigate the relevant equation in outer boundary layers. When $\alpha^{\frac{2}{3}} \gg \epsilon$, it is found that the mean tangential velocity is $O(\alpha^{\frac{4}{3}})$ near the interface, and that decay to zero takes place through outer interfacial boundary layers of thickness $O(\epsilon/\alpha^{\frac{2}{3}})$. In practice, such velocities can be a significant fraction of the primary, oscillatory velocity. Near the interface, 'jet-like' motions are predicted in the neighbourhood of *nodal* positions, and arise because of the collision of mean flows, in *both outer and inner* interfacial boundary layers, in adjacent cells of width one-quarter of a wavelength. In the time-averaged picture, the interface is horizontal and the axes of the jets are vertical. The momentum flux in each outer boundary layer is obtained analytically at any position along the interface, and the ratio of the fluxes is equal to the ratio of the corresponding quantities $(\rho\mu)^{\frac{1}{2}}$, where ρ and μ denote the density and viscosity of a fluid, respectively.

2. Summary of existing results

We consider the case of two-dimensional motion associated with a standing wave on the interface between two homogeneous, immiscible, incompressible fluids. The origin of Cartesian co-ordinates is taken at the equilibrium level of the interface, and the density $\rho^{(1)}$ of the lower fluid is assumed to be greater than that, $\rho^{(2)}$, of the upper fluid. The lower fluid is assumed either to have infinite depth or to be bounded below by the fixed horizontal plane $z = -h^{(1)}$. The upper fluid may either have infinite depth, be bounded by the fixed plane $z = h^{(2)}$ or possess a free upper surface. If, in the latter case, $\eta \equiv (\rho^{(1)} - \rho^{(2)})/\rho^{(1)} \ll 1$, interest is largely confined to the internal mode; for larger values of η, each of the two possible modes is of interest. The standing wave to be considered has period $2\pi/\sigma$ and wavelength $2\pi/k$. A stream function ψ is defined such that the velocity vector $\mathbf{q} = (u, w) = (\partial\psi/\partial z, -\partial\psi/\partial x)$, and the variables are non-dimensionalized according to the scheme

$$\hat{\mathbf{r}} = k\mathbf{r}, \quad \hat{t} = \sigma t, \quad \hat{\psi} = k^2\psi/\sigma, \quad \hat{\epsilon} = (\nu k^2/\sigma)^{\frac{1}{2}},$$

where ν denotes the kinematic viscosity. Then, omitting the carets, $\omega = \nabla^2\psi$ represents the fluid vorticity and the interfacial displacement is

$$z_i = \alpha \cos x e^{it} + O(\alpha^2) \quad (\alpha \ll 1). \tag{2.1}$$

Boundary-layer theory is assumed to be applicable in each fluid, so that we necessarily require that

$$\epsilon^{(r)} \ll \min(1, h^{(r)}) \quad (r = 1, 2), \tag{2.2}$$

where superscripts (1) and (2) denote quantities associated with the lower and upper fluid, respectively. In order to investigate the interfacial boundary layers, we shall make use of the orthogonal curvilinear co-ordinate system described by Longuet-Higgins (1953). Thus s denotes arc length measured along the interface and n is measured positive along a normal into the *upper* fluid; $\kappa(s, t)$ denotes the curvature of the interface and is positive when the centre of curvature lies in the upper fluid. The quantities s, n and κ are used below, following their non-dimensionalization according to the above-mentioned scheme.

Within the oscillatory boundary layers of thickness $O(\epsilon)$ adjacent to the interface, which is assumed to be uncontaminated, we write

$$N = n/2^{\frac{1}{2}}\epsilon, \quad \psi = \alpha\psi_{(1)}(s, N, t) + \alpha^2\psi_{(2)} + \dots. \tag{2.3}$$

Although $\psi_{(1)}$ is $O(1)$, the order of magnitude of $\psi_{(2)}$ cannot be determined without consideration of the mean flow outside these layers. Regarding the mean motion within the layers, Dore (1973)† has shown that

$$[\mu\overline{\omega}_{(2)}] = \mu^{(1)}\overline{\omega}^{(1)}_{(2)}\Big|_{N^{(1)}_\infty} - \mu^{(2)}\overline{\omega}^{(2)}_{(2)}\Big|_{N^{(2)}_\infty}$$

$$\approx \frac{1+i}{2^{\frac{3}{2}}i}\frac{\nu^{\frac{1}{2}}}{\epsilon}\frac{(\rho^{(1)}\rho^{(2)}\mu^{(1)}\mu^{(2)})^{\frac{1}{2}}}{(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} + (\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}}\Delta\frac{\partial\Delta^*}{\partial s}, \tag{2.4}$$

$$[\overline{q}^{(2)}_s] = O(1), \tag{2.5}$$

where μ is the viscosity of a fluid, a bar denotes an average over a wave period and $\Delta = [q_{s(1)}]$ denotes the change in the tangential component of velocity across the whole region of the oscillatory interfacial boundary layers. That is, Δ represents the strength of the interfacial vortex sheet according to linear, inviscid theory. These results form boundary conditions on the mean motion outside the oscillatory layers, and were obtained by integrating the mean equation of motion across the layers and satisfying the interfacial conditions of continuity of

† In relation to this work, it may be noted that the right-hand side of equation (A 24) of the appendix should be multiplied by a factor $2^{-\frac{1}{2}}$; equation (3.10) of the work is, however, accurate.

tangential velocity and tangential stress. (It is noted that the right-hand side of (2.4) vanishes when $\rho^{(2)} = 0$ or when $\rho^{(1)} = \rho^{(2)}$.) It was found by Dore (1973) that the order of magnitude of the mean vorticity both within and at the edges of the oscillatory layers is *at least* $\alpha^2\epsilon^{-1}$. In fact, we assume that $\overline{\omega}_{(2)}$ is $O(\epsilon^{-1})$ throughout these layers. The order of magnitude of the mean tangential velocity within the layers is then *at least* α^2, but if it is greater than this, the *main* contribution to the mean tangential velocity must be constant across any section s = constant through the layers.

For standing interfacial waves of sufficiently small amplitude that $1 \gg \epsilon \gg \alpha^{\frac{2}{3}}$, t was shown by Dore (1973) that the mean motion outside the oscillatory layers† is governed by the biharmonic equation for $\overline{\Psi}^{(r)}_2(x,z)$ and that mean vorticity $O(\alpha^2\epsilon^{-1})$ diffuses into the interior of each fluid. This results in a mean circulation of velocity $O(\alpha^2\epsilon^{-1})$ in cells of width one-quarter of a wavelength. Correspondingly, the leading term in the function $\overline{\psi}^{(r)}_{(2)}$ associated with the oscillatory interfacial layers is of the form $\overline{\psi}_{(20)} = C(s)N$, where $C(s)$ is $O(1)$, so that the mean tangential velocity within these layers is $O(\alpha^2\epsilon^{-1})$, and is constant across them, in accordance with remarks made in the previous paragraph. We now investigate the mean motion for standing interfacial waves of sufficiently large amplitude that $\alpha^{\frac{2}{3}} \gg \epsilon$, and make some use of the theory of Stuart (1963, 1966) and Riley (1965, 1967).

3. Double boundary layers for the case $\epsilon \ll \alpha^{\frac{2}{3}} \ll 1$

In discussing mean motions associated with interfacial waves, the interfacial boundary layers are of prime importance and, as mentioned in § 2, the mean vorticity, which is $O(\alpha^2\epsilon^{-1})$, does not decay to zero at the edges of the oscillatory layers. We therefore consider the possibility of the existence of outer interfacial boundary layers, through which such vorticity, together with the associated mean tangential velocity from which it is derived, may decay to zero. Because of the fluctuating position of the interface and inner interfacial layers, and because of the rapid variations expected within the outer layers, it is necessary to use a system of curvilinear co-ordinates in which the interface coincides with a co-ordinate surface. The system which we adopt is that of Longuet-Higgins (1953), referred to in § 2. In terms of these co-ordinates (s, n), it is readily shown (see appendix) that the dominant contribution to the mean vorticity satisfies the nonlinear partial differential equation

$$Q_s \frac{\partial \overline{\omega}_{(2)}}{\partial s} + Q_n \frac{\partial \overline{\omega}_{(2)}}{\partial n} = \frac{\epsilon^2}{\alpha^2} \nabla^2 \overline{\omega}_{(2)}, \tag{3.1}$$

where

$$\left.\begin{aligned} Q_s &= \overline{s}'_{(2)} + \overline{\int s'_{(1)}\, dt\, \partial s'_{(1)}/\partial s} + \overline{\int n'_{(1)}\, dt\, \partial s'_{(1)}/\partial n}, \\ Q_n &= \overline{n}'_{(2)} + \overline{\int s'_{(1)}\, dt\, \partial n'_{(1)}/\partial s} + \overline{\int n'_{(1)}\, dt\, \partial n'_{(1)}/\partial n} \end{aligned}\right\} \tag{3.2}$$

† In the interior of the fluid, $\psi = \alpha\Psi_1 + \alpha^2\Psi_2 + \dots$.

denote the components of mass transport velocity along and perpendicular to the interface, respectively, and s' and n' represent the rates at which the co-ordinates of a particular fluid element are increasing. For a single standing wave, the integrals in (3.2) yield no $O(1)$ contributions, so that we expect that $\overline{\omega}_{(2)}$ satisfies

$$\overline{s}'_{(2)}\frac{\partial\overline{\omega}_{(2)}}{\partial s}+\overline{n}'_{(2)}\frac{\partial\overline{\omega}_{(2)}}{\partial n}=\frac{\epsilon^2}{\alpha^2}\nabla^2\overline{\omega}_{(2)}. \tag{3.3}$$

Also, within the outer layers, where mean inertia and viscous forces are comparable, we expect that $\overline{\omega}_{(2)}$ is $O(\epsilon^{-1})$ and that the mean tangential quantity $\overline{s}'_{(2)} \approx \partial\overline{\Psi}_{(2)}/\partial n$ must be $O(\delta\epsilon^{-1})$, where δ is a measure of the thickness of the outer interfacial boundary layers. (In fact, the possibility that $\overline{s}'_{(2)}$ can be greater than $O(\delta\epsilon^{-1})$, and hence constant across a section s = constant, can be shown to be untenable.) Thus, from consideration of orders of magnitude in (3.3), we obtain

$$\delta = \epsilon/\alpha^{\frac{2}{3}} \ll 1, \tag{3.4}$$

and define

$$N = n/\delta, \quad \overline{\Psi}_{(2)} = (\delta/\alpha^{\frac{2}{3}})\,\Phi(s,N)+\dots \tag{3.5}$$

within the outer layers. Consequently, regarding the mass transport velocity outside the oscillatory layers, it is the parameter $\alpha^{\frac{4}{3}}/\epsilon^2$ which is the analogue of the conventional Reynolds number. The mean tangential velocity in the outer layers is $O(\alpha^{\frac{1}{3}})$, which, although surprisingly large, remains (formally) much less than the oscillatory velocity, $O(\alpha)$, of the primary motion. The expansion of $\overline{\psi}_{(2)}(s,N)$ in (2.3) begins with a term of the form $\delta C(s)N$, the corresponding mean tangential velocity, $O(\alpha^{\frac{1}{3}})$, being constant across the inner interfacial boundary layers.

On using (3.3)–(3.5), together with the assumption that the mean tangential velocity, $O(\alpha^{\frac{1}{3}})$, decays to zero at the edges of the outer interfacial boundary layers, we obtain an equation of the form

$$\frac{\partial\Phi}{\partial N}\frac{\partial^2\Phi}{\partial s\,\partial N}-\frac{\partial\Phi}{\partial s}\frac{\partial^2\Phi}{\partial N^2}=\frac{\partial^3\Phi}{\partial N^3} \tag{3.6}$$

in each of the outer interfacial layers. Thus the main contribution to the mean motion in these layers is governed by an equation of the same structure as that found by Dore (1976) to describe the mean flow in the outer bottom boundary layer beneath a standing surface wave. The boundary conditions to be satisfied by Φ are determined with the aid of (2.4) and (2.5), the kinematical condition at the interface and principles of asymptotic matching:

$$\frac{\mu^{(1)}}{\epsilon^{(1)}}\frac{\partial^2\Phi^{(1)}}{\partial N^{(1)2}}-\frac{\mu^{(2)}}{\epsilon^{(2)}}\frac{\partial^2\Phi^{(2)}}{\partial N^{(2)2}}=\frac{K}{2^{\frac{3}{2}}}\frac{\nu^{\frac{1}{2}}}{\epsilon}\frac{(\rho^{(1)}\rho^{(2)}\mu^{(1)}\mu^{(2)})^{\frac{1}{2}}}{(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}}+(\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}}\sin 2s \quad (N=0), \tag{3.7}$$

$$\partial\Phi^{(1)}/\partial N^{(1)}-\partial\Phi^{(2)}/\partial N^{(2)}=0 \quad (N=0), \tag{3.8}$$

$$\Phi^{(1)}=\Phi^{(2)}=0 \quad (N=0) \tag{3.9}$$

and

$$\partial\Phi^{(1)}/\partial N^{(1)},\ \partial\Phi^{(2)}/\partial N^{(2)}\to 0 \tag{3.10}$$

at the edges of the respective outer boundary layers. It is important to note that $K > 0$ for each motion in the form of a single standing wave. Specifically,

$$K = \tfrac{1}{2}(\coth h^{(1)} + \coth h^{(2)})^2$$

when the uppermost surface of the system is the plane $z = h^{(2)}$,

$$K = \tfrac{1}{2}[(\coth h^{(1)} + \coth h^{(2)}) - (\beta/\alpha)\,\mathrm{cosech}\, h^{(2)}]^2$$

when the uppermost surface is free and has amplitude β. In the latter case, the formula for K holds irrespective of which of the two possible modes of oscillation forms the standing wave. Physically, (3.7) and (3.10) imply that there is a (net) mean tangential stress $O(\alpha^2\epsilon)$ acting on the fluid within the outer boundary layers. This stress is exerted in the positive (negative) s direction between $s = r\pi$ ($s = r\pi - \frac{1}{2}\pi$) and $s = r\pi + \frac{1}{2}\pi$ ($s = r\pi + \pi$), $r = 0, \pm 1, \pm 2, \ldots$. Consequently, it may be expected that the mean flows in the outer boundary layers converge in the regions of the nodes. Therefore, to the above conditions on Φ, we add the requirement that

$$\partial\Phi/\partial N = 0 \quad (s = r\pi). \tag{3.11}$$

When (3.6) is integrated across the whole of each outer boundary layer and use is made of the conditions (3.7), (3.9) and (3.10), we find the following analytical relation for the total momentum flux:

$$M^{(1)}(s) + M^{(2)}(s) = \frac{K\alpha^2\epsilon}{2^{\frac{3}{2}}\nu^{\frac{1}{2}}} \frac{(\rho^{(1)}\rho^{(2)}\mu^{(1)}\mu^{(2)})^{\frac{1}{2}}}{(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} + (\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}} \sin^2 s,$$

where

$$M^{(1)}(s) = \alpha^{\frac{8}{3}}\delta^{(1)}\rho^{(1)} \int_{-\infty}^{0} \left(\frac{\partial\Phi^{(1)}}{\partial N^{(1)}}\right)^2 dN^{(1)},$$

$$M^{(2)}(s) = \alpha^{\frac{8}{3}}\delta^{(2)}\rho^{(2)} \int_{0}^{\infty} \left(\frac{\partial\Phi^{(2)}}{\partial N^{(2)}}\right)^2 dN^{(2)},$$

However, the above boundary-value problem for Φ may, with some advantage, be transformed as follows. We write

$$\Phi^{(1)}(s, N^{(1)}) = \gamma F(s, \gamma N^{(1)}), \quad \Phi^{(2)}(s, N^{(2)}) = -\gamma F(s, -\gamma N^{(2)}), \tag{3.12}$$

where

$$\gamma = \left[\frac{K}{2^{\frac{3}{2}}} \frac{(\rho^{(1)}\rho^{(2)}\mu^{(1)}\mu^{(2)})^{\frac{1}{2}}}{[(\rho^{(1)}\mu^{(1)})^{\frac{1}{2}} + (\rho^{(2)}\mu^{(2)})^{\frac{1}{2}}]^2}\right]^{\frac{1}{3}}. \tag{3.13}$$

Then the function $F(s, N)$, defined for $-\infty < N \leqslant 0$, is the solution of the boundary-value problem

$$\frac{\partial F}{\partial N}\frac{\partial^2 F}{\partial s\,\partial N} - \frac{\partial F}{\partial s}\frac{\partial^2 F}{\partial N^2} = \frac{\partial^3 F}{\partial N^3},$$

$$F = 0 \quad (N = 0),$$

$$\partial^2 F/\partial N^2 = \sin 2s \quad (N = 0),$$

$$\partial F/\partial N \to 0 \quad (N \to -\infty),$$

$$\partial F/\partial N = 0 \quad (s = r\pi).$$

With reference to the functions $\Phi^{(1)}(s, N^{(1)})$ and $\Phi^{(2)}(s, N^{(2)})$ associated with the outer interfacial boundary layers, the value of the above transformation is apparent. These functions, which satisfy the same partial differential equation and which are linked by the 'interfacial' boundary conditions (3.7) and (3.8), may, for any values of $\rho^{(r)}$, $\mu^{(r)}$ and $h^{(r)}$ and whether the uppermost surface is free or not, be inferred directly from the solution of the boundary-value problem for $F(s, N)$. In particular, velocity profiles $\alpha^2 \bar{s}'_{(2)} \approx \alpha^{\frac{4}{3}}\, \partial\Phi/\partial N$ at any section $s =$ constant in the respective outer boundary layers have a similar form. Further,

$$\int_{-\infty}^{0} \left(\frac{\partial F}{\partial N}\right)^2 dN = \sin^2 s, \tag{3.14}$$

so that

$$M^{(r)}(s) = \alpha^{\frac{8}{3}} \delta^{(r)} \rho^{(r)} \gamma^3 \sin^2 s \tag{3.15}$$

may be determined simply and exactly, and shows that the ratio of the momentum fluxes in the outer layers is equal to the ratio of the corresponding quantities $(\rho\mu)^{\frac{1}{2}}$. Each such flux increases monotonically from anti-nodal to nodal positions, and reaches a (theoretical) maximum at nodal locations $s = (r+\frac{1}{2})\pi$. The 'terminal' momentum flux arriving at any node is given by

$$2M^{(r)}(r\pi + \tfrac{1}{2}\pi) = 2\alpha^{\frac{8}{3}} \delta^{(r)} \rho^{(r)} \gamma^3 \tag{3.16}$$

in each outer boundary layer.

With reference to the boundary-value problem for $F(s, N)$, we shall be content here to develop an approximate solution in the neighbourhood of anti-nodes $s = r\pi$. Thus we adapt the procedure of Riley (1965) and seek a solution in terms of a series expansion, analogous to the Blasius series, about the anti-nodal stagnation points of the mean flow in the outer interfacial boundary layers. We write

$$\tilde{s} = s - r\pi, \quad \sin 2\tilde{s} = c_1^3 \tilde{s} + c_1^2 c_3 \tilde{s}^3 + c_1^2 c_5 \tilde{s}^5 + \ldots,$$

$$F(\tilde{s}, N) = c_1 \tilde{s} f_1(\hat{N}) + c_2 \tilde{s}^3 f_3(\hat{N}) + \ldots,$$

where $\hat{N} = c_1 N$ $(c_1 > 0)$, and obtain

$$f_1 = e^{\hat{N}} - 1,$$

$$f_3 = -\frac{7}{146} e^{\hat{N}} - \frac{6}{73}\left[1 - 3\hat{N} e^{\hat{N}} - \frac{3}{2} e^{2\hat{N}} - \frac{1}{12} e^{3\hat{N}}\right].$$

Thus, for $|\tilde{s}| \ll 1$, profiles of the mean tangentia lvelocity $\partial_{(2)}\bar{\Psi}/\partial N$ are approximately exponential. Within the inner oscillatory boundary layers, the main contribution to the mean tangential velocity is $O(\alpha^{\frac{4}{3}})$, and is constant across any section $s =$ constant, as mentioned above. In fact, we have

$$Q_s \approx \bar{s}'_{(2)} \approx \alpha^{\frac{4}{3}} (\partial\Phi/\partial N)_{N=0} = \alpha^{\frac{4}{3}} c_1 \gamma^2 [c_1 \tilde{s} f_1'(0) + c_3 \tilde{s}^3 f_3'(0) + \ldots]$$

within these layers. By consideration of (3.6) and the requirement that the mean tangential velocity, $O(\alpha^{\frac{4}{3}})$, should tend to zero at the edges of the outer layers, it is readily shown that the mean normal velocity $Q_n \approx \bar{n}_2'$, which is $O(\alpha^{\frac{4}{3}}\delta)$, must be negative (positive) at the edge of the outer interfacial boundary layer in the upper (lower) fluid.

4. Description of the mean flow

Features of the mean Lagrangian flow pattern within the interfacial boundary layers are as follows. Fluid elements in any particular cell, of width one-quarter of a wavelength, move steadily, in *both inner and outer* layers, in the direction from anti-nodes to nodes; conservation of mass is maintained, since fluid at the edges of the outer layers moves normally inwards. The mean flows in the layers collide in the neighbourhood of nodal positions. It can then be argued, as for the outer bottom boundary layer beneath a standing surface wave (considered by Dore 1976), that the present results may indicate the occurrence of a sequence of 'jet-like' motions occurring in the neighbourhood of *nodal* positions, the strength of the 'jets' being represented by (3.16). Thus it is of interest to compare the steady velocities in the outer layer beneath a standing surface wave with those $[O(\alpha^{\frac{4}{3}})]$ in the interfacial layers. If, as a particular example, we take $\eta = 10^{-3}$ and have the same wavelength in the two cases, the (dimensional) steady velocities are comparable over the *whole* range of surface/interfacial slopes and, correspondingly, the ratio of surface to interfacial wave amplitudes is $O(1)$. Therefore it would seem that use of the terminology 'jet' can be equally justified in the present context for sufficiently large α. Accordingly, in the time-averaged picture of the flow, wherein the interface is *horizontal*, thin jets emerge from nodal positions and have axes directed *vertically* upwards (downwards) in the upper (lower) fluid.

If, in the solution of § 3, the depth of each fluid is infinite, the double-boundary-layer theory requires that

$$\delta = \epsilon/\alpha^{\frac{2}{3}} \ll 1.$$

For fluids of finite depth, the corresponding condition is

$$\delta \ll \min(1, h),$$

and there are two principal cases $\alpha \gtrless \epsilon$.

(*a*) $\epsilon \ll \alpha \ll \alpha^{\frac{2}{3}} \ll 1$. For interfacial amplitudes satisfying this condition, an outer boundary layer of thickness $O(\epsilon/\alpha)$ is present near the rigid horizontal bottom. Within this layer, the structure of the mean motion is similar to that described for the outer boundary layer beneath a standing surface wave, and a horizontal mass transport velocity $O(\alpha^2)$ decays to zero at the edge of the layer. A sequence of vertical jets is, in general, present near the bottom, beneath anti-nodal positions.† In any particular cell of width one-quarter of a wavelength, the jets at the interface and near the bottom are such as to tend to produce interior circulatory flow in the same sense. Of course, the strength of the interfacial jets greatly exceeds that of the bottom jets, by a factor $O(\alpha^{-1})$.

(*b*) $\alpha \ll \epsilon \ll \alpha^{\frac{2}{3}} \ll 1$. In this case, the only double boundary layers present occur near the interface. The mean flow near the bottom, but outside the oscillatory layer, is dominated by diffusion of mean vorticity $O(\alpha^2)$. This would tend to produce mass transport velocities $O(\alpha^2)$, which are, however, much smaller than those in the interfacial boundary layers and jets.

† But, in the important case when $\eta \ll 1$, the vertical motion at anti-nodal positions is likely to be very weak, and the terminology 'jet' may not be appropriate.

For cases of finite depth, certain qualifying remarks must be made concerning each amplitude range (*a*) and (*b*). That an oscillating clean free surface does not give rise to a sequence of downward almost-vertical jets has been shown by Dore (1976). In the present context, this must be subject to the proviso that the jets emerging from the interface have no significant effect near the free surface. This condition should be satisfied if $\alpha^{\frac{3}{2}}/\epsilon$ is not too large and the waves are sufficiently short. The mean circulation pattern in the upper fluid should then be confined to the lower part of that fluid. Also, if one or both of the fluids is bounded by a rigid horizontal boundary, it is assumed that the effect of the interfacial jets near such a boundary is insignificant.

The author is grateful to a referee for helpful comments which have led to an improved § 4.

Appendix

Brief details are given of the derivation of the partial differential equation which governs the mean motion in outer boundary layers adjacent to an interface. Use is made of the orthogonal curvilinear co-ordinate system (s, n) described by Longuet-Higgins (1953).

The vorticity equation for two-dimensional motion is

$$\left(\frac{\partial}{\partial t}+s'\frac{\partial}{\partial s}+n'\frac{\partial}{\partial n}\right)\omega = \epsilon^2\nabla^2\omega, \tag{A 1}$$

$$\nabla^2 \equiv \frac{1}{\zeta}\left[\frac{\partial}{\partial s}\left(\frac{1}{\zeta}\frac{\partial}{\partial s}\right)+\frac{\partial}{\partial n}\left(\zeta\frac{\partial}{\partial n}\right)\right] \quad (\zeta = 1-n\kappa),$$

with $\alpha \ll 1$. We write

$$\omega = \alpha\omega_{(1)}+\alpha^2\omega_{(2)}+\ldots, \tag{A 2}$$

with similar expansions for s' and n'. Just outside the oscillatory layers, $\omega_{(1)} = 0$ and $\overline{\omega}_{(2)}$ is expected to be $O(\epsilon^{-1})$, but the order of magnitude of $\overline{s}'_{(2)}$ is, *a priori*, neither known nor immediately inferable. At first, we assume that all derivatives are $O(1)$. Then the most important terms in (A 1) yield

$$\partial\omega_{(2)}/\partial t = 0, \quad \omega_{(2)} = \overline{\omega}_{(2)}. \tag{A 3}$$

From consideration of the next most important unsteady terms, and the fact that $s' = O(\alpha)$ and $n' = O(\alpha\epsilon)$ at the edges of the inner layers, we conclude that

$$\omega_{(3)} = -\int s'_{(1)}\,dt\,\frac{\partial\overline{\omega}_{(2)}}{\partial s}-\int n'_{(1)}\,dt\,\frac{\partial\overline{\omega}_{(2)}}{\partial n}+\overline{\omega}_{(3)}. \tag{A 4}$$

On taking the mean parts of the next most important terms, we find that the dominant contribution to the mean vorticity $\overline{\omega}_{(2)}$ satisfies

$$Q_s\frac{\partial\overline{\omega}_{(2)}}{\partial s}+Q_n\frac{\partial\overline{\omega}_{(2)}}{\partial n} = \frac{\epsilon^2}{\alpha^2}\nabla^2\overline{\omega}_{(2)}, \tag{A 5}$$

as discussed in § 3.

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